EQUILIBRE DE NASH & TRANSPORT OPTIMAL

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Modelisation with optimal transport (ANR TOMMI)

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GAME THEORY

The players choose actions in a given set. The payoff of the agent i depends on her action a_i and the actions of all the other players a_{-i}. We denote the payoff $\Pi(a_i, a_{-i})$.
A player can also play in *mixed strategy*, *i.e.* to play a strategy e_j with a probability x_j. This mixed strategy is thus given by a vector $(x_1, \cdots, x_N)$.
If the strategy y of Player 2 is known we say that Player 1 is in *best reply* against y if her action x^* is such that

$$x^* = \mathrm{Argmax}_x \Pi(x, y) \; .$$

A pair (x, y) is a *Nash equilibirum* if each agent is in best reply against the other player's action (*i.e.* all the agents have no incentive to relocate).

NASH (1950)

"*The theory of non-cooperative games is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration and communication from any of the others.* "

$\rightarrow$ Existence of equilibria in a non-cooperative n-persons game ($n \in \mathbb{N}$).

VON NEUMANN-MORGENSTERN (1944)

"*An almost exact theory of a gas, containing about* 10^{25} *freely moving particles, is incomparably easier than that of the solar system, made up of 9 major bodies.*"

"*It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case.*"

"*When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory become possible.*"

SCHMEIDLER (1973)

"*Non-atomic games enable us to analyze a conflict situation where the single player has no influence on the situation.*"

$\rightarrow$ Existence of an equilibria in a non-atomic game with an arbitrary finite number of pure strategies. See also [Mas-Colell, 1984].

Consider a *non-cooperative anonymous game with a continuum of agents* (= “mean field game” in Pierre-Louis Lions’ terminology).

COST FUNCTION

The agent has to take action in a compact metric action space Y. Given an action distribution $\nu \in \mathcal{P}(Y)$ the agent taking action y incurs the cost

$$\Pi(y, \nu) := \mathcal{V}[\nu](y) \ .$$

RIVALRY/CONGESTION

The utility of the agent decreases when the number of players who choose the same action increases.

Examples:

- Consumption of the same public good (motorway game),
- Food supply in an habitat decreases with the number of its users (ex. Sticklebacks (Milinsky)).

INTERACTIONS

The utility of the agents increases because some other agents play a similar action.

Examples:

- Location to go shopping,
- Quality of a product in a differentiated industry.

EXTERNALITIES IN [BECKMANN, 1976]'S MODEL

- Congestion: the agents benefit from social interactions but there is a cost to access to distant agents,
- Interaction: more populated areas lead to higher competition for land.

Let $\mathcal{K}$ be a convex domain of $\mathbb{R}^d$ and ν the density of agents. We assume that ν is a probability density.

INDIRECT COST FUNCTIONAL

Consider

$$\mathcal{V}[\nu](y) := \underbrace{f[\nu(y)]}_{\text{congestion}} + \underbrace{\int_{\mathcal{K}} \phi(|y-z|)\nu(z)\,\mathrm{d}z}_{\text{interaction}} + \underbrace{A(y)}_{\text{amenities}} \quad .$$

where

- f is the competition for land. We assume that f is an increasing function.
- ϕ is the travelling cost. We assume that ϕ is a non-negative and radially symmetric continuous function.
- A is an external potential. We assume that A is a continuous function bounded from below.

NASH EQUILIBRIUM

The probability $\nu \in \mathcal{P}(Y)$ is a Nash equilibrium if:

$$\begin{cases} \mathcal{V}[\nu](y) = V & \nu\text{-a.e. } y, \\ \mathcal{V}[\nu](y) \geq V & \text{a.e. } y \in Y. \end{cases}$$

Consider now that the agents have a given type x in a compact metric space X. Given an action distribution $\nu \in \mathcal{P}(Y)$, the type-x agent taking action y incurs the cost

$$\Pi(x, y, \nu) \; .$$

Assume

COST IN A SEPARABLE FORM

$$\Pi(x, y, \nu) := c(x, y) + \mathcal{V}[\nu](y) \; .$$

NASH EQUILIBRIUM

The probability $\gamma \in \mathcal{P}(X \times Y)$ is a Nash equilibrium if:

- its first marginal is μ ,
- its second marginal ν is such that there exists a function φ such that

$$\begin{cases} \Pi(x, y, \nu) = \varphi(x) & \gamma\text{-a.e. } (x, y), \\ \Pi(x, y, \nu) \geq \varphi(x) & \text{a.e. } (x, y) \in X \times Y. \end{cases}$$

EXISTENCE AND UNIQUENESS [B., MOSSAY & SANTAMBROGIO, 2012] AND [B. & CARLIER, 2012]

There exists a unique Nash equilibrium.

Our results apply to

POTENTIAL GAMES (SEE [MONDERER-SHAPLEY, 1996] FOR A FINITE NUMBER OF PLAYERS)

There exists a functional $\mathcal{E}$ such that $\mathcal{V}[\nu]$ is the first variation of $\mathcal{E}$ *i.e.*

$$\mathcal{V}[\nu] = \frac{\delta \mathcal{E}}{\delta \nu} .$$

Under the assumptions:

- $\mathcal{E}$ displacement convex and coercive. Ex.: ϕ convex symmetric and the congestion function satisfies the Inada condition.
- c satisfies a generalised Spence-Mirrlees condition *i.e.* for every x, $y \mapsto \nabla_x c(x, y)$ is injective. Ex. c smooth and strictly convex.

For sake of simplicity, we assume from now on

$$c(x, y) = \frac{|x - y|^2}{2} .$$

CONNECTION WITH OPTIMAL TRANSPORT

Let $\gamma \in \mathcal{P}(X \times Y)$ be a Nash equilibrium of second marginal ν. Then γ is a solution to the Kantorovich problem, *i.e.* γ is a solution to

$$\min_{\Pi_X\gamma=\mu, \Pi_Y\gamma=\nu} \iint_{X\times Y} c(x,y)\,\mathrm{d}\gamma(x,y) =: \mathrm{W}_c(\mu,\nu)$$

Proof: Let η be of first marginal μ and second marginal ν then we have

$$\begin{aligned}\iint_{X\times Y} c(x,y)\,\mathrm{d}\eta(x,y) &\geq \iint_{X\times Y} (\varphi(x) - V[\nu](y))\,\mathrm{d}\eta(x,y) \\ &= \int_X \varphi(x)\,\mathrm{d}\mu(x) - \int_Y V[\nu](y)\,\mathrm{d}\nu(y) = \iint_{X\times Y} c(x,y)\,\mathrm{d}\gamma(x,y)\,.\end{aligned}$$

PURITY OF THE EQUILIBRIUM

If μ does not give weight to points then any Nash equilibrium is pure.

VARIATIONAL PROBLEM

$$\inf_{\nu \in \mathcal{P}(Y)} \left\{ \frac{1}{2} \mathcal{W}_2^2(\mu, \nu) + \mathcal{E}[\nu] \right\} \tag{1}$$

where

$$\mathcal{E}[\nu] = \int_{\mathcal{K}} F(\nu(x))\, \mathrm{d}x + \int_{\mathcal{K}} A(x)\, \mathrm{d}\nu + \frac{1}{2} \iint_{\mathcal{K}^2} \phi(|x-y|)\nu(x)\nu(y)\, \mathrm{d}x\, \mathrm{d}y\ .$$

andwhere F is an antiderivative of f and the Monge-Kantorovich distance is defined by

$$\mathrm{W}_2^2(\mu, \nu) := \min_{\Pi_X \gamma = \mu, \Pi_Y \gamma = \nu} \iint_{X \times Y} \frac{|x-y|^2}{2}\, \mathrm{d}\gamma(x, y)$$

EQUIVALENCE BETWEEN EQUILIBRIUM AND MINIMISER

$\gamma \in \mathcal{P}(X \times Y)$ is a Nash equilibrium if and only if

- ν is a minimiser of (1),
- γ is a solution to the Kantorovich problem.

Let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. A measurable function $T : X \to Y$ *pushes-forward* μ onto ν, and we denote $T\#\mu = \nu$, if

$$\forall \zeta \in \mathcal{C}_b^0(Y), \int_X \zeta\,[T(x)]\,\mathrm{d}\mu(x) = \int_Y \zeta(y)\,\mathrm{d}\nu(y)\,.$$

Brenier's theorem (CPAM, 1991)

There exists a unique optimal transport map T solution to the Kantorovich problem. Moreover it is a solution to the Monge problem

$$\inf_{T:T\#\mu=\nu} \int_X |x - T(x)|^2\,\mathrm{d}\mu(x) = \mathrm{W}_2^2(\mu,\nu)\,.$$

DISPLACEMENT INTERPOLATION, SEE MCCANN (ADV. MATH., 1997)

Let T be the optimal transport map which transports $\rho_0\,\mathrm{d}x$ onto $\rho_1\,\mathrm{d}y$. The *displacement interpolation* between ρ_0 and ρ_1 is

$$\rho_t = [(1-t)\mathrm{id} + tT]\,\#\rho_0\ .$$

DISPLACEMENT CONVEXITY

A functional $\mathcal{G}$ is *displacement convex* if for all $\rho_0 \in \mathcal{P}(X)$, $\rho_1 \in \mathcal{P}(Y)$

$$\mathcal{G}[\rho_t] \leq (1-t)\mathcal{G}[\rho_0] + t\mathcal{G}[\rho_1].$$

CRITERIA OF DISPLACEMENT CONVEXITY, SEE MCCANN (ADV. MATH., 1997)

Assume that

- $F(0) = 0$ and $r \mapsto r^d F(r^{-d})$ is convex non-increasing,
- A is convex then $\mathcal{V}[\nu]$ is displacement convex.
- ϕ is convex then $\mathcal{W}[\nu]$ is displacement convex.

then $\mathcal{E}$ is displacement convex.

A partial differential equation for the equilibrium

Let u be a solution to the following Monge-Ampère equation

$$\mu(x) = \det(D^2 u(x)) \exp\left(-\frac{|\nabla u(x)|^2}{2} + x \cdot \nabla u(x) - u(x) - \int_Y \phi(\nabla u(y), \nabla u(z))\, \mathrm{d}\mu(z)\right)$$

then $\varphi(x) = u(x) + |x|^2/2$ is the optimal transport which transport μ onto ν so that $\nu = \varphi\#\mu$.

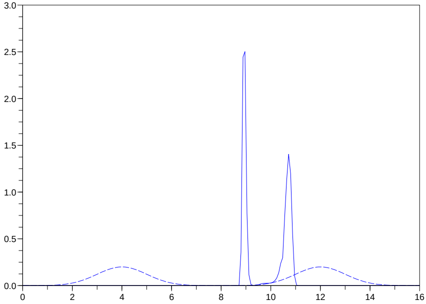

FIGURE: The distribution μ in dash line and ν in the case $f(x) = x^8$, $\phi(z) = 10^{-4}|z|^2$ and $A = (x - 10)^4$.

SOCIAL WELFARE

$$\iint_{X\times Y} \Pi(x,y,\nu)\,\mathrm{d}\gamma = \iint_{X\times Y} \frac{|x-y|^2}{2}\,\mathrm{d}\gamma + \int_Y \left[f[\nu(y)] + \int_Y \phi(y,z)\,\mathrm{d}\nu(z) + A(y) \right] \mathrm{d}\nu(y).$$

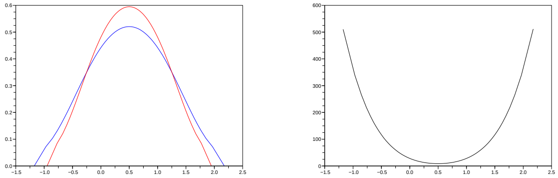

FIGURE: Left: the optimum (red) and the equilibrium (blue). Right: tax at the equilibrium. Cost of anarchy ~ 1.8.

TAX TO RESTORE EFFICIENCY

$$\mathrm{Tax}[\nu](y) = \nu(y) f[\nu(y)] - F[\nu(y)] + \frac{1}{2}\int_Y \phi(y,z)\,\mathrm{d}\nu(z)\ .$$

The agents start with a distribution of strategies and adjust it over time by choosing

MINIMISING SCHEME

$$\nu_{k+1} \in \operatorname{argmin}_\nu \left\{ \frac{1}{2\tau} \mathcal{W}_2^2(\nu_k, \nu) + \mathcal{E}[\nu] \right\} .$$

This scheme converges in some sense to the

CONTINUOUS EVOLUTION EQUATION

$$\frac{\partial \nu}{\partial t} + \operatorname{div}\left(-\nu \nabla \mathcal{V}[\nu]\right) = 0,$$

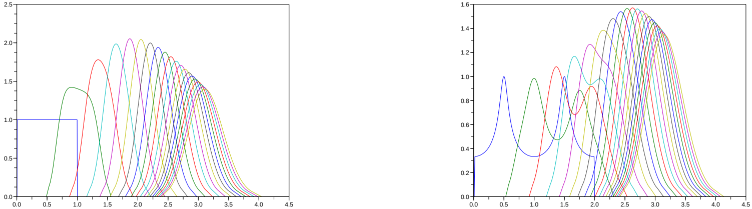

FIGURE: Convergence and stabilisation toward the equilibrium in the case of a logarithmic congestion, cubic interaction, and a potential $A(x) := (x-5)^3$ with $\mathbb{1}_{[0,1]}$ as initial guess (left) and made of two bumps (right).

Merci pour votre attention