From Brenier to Kntohe and from Knothe to Brenier: convergence, PDE and numerical ideas

Filippo Santambrogio

Laboratoire de Mathématiques d'Orsay, Université Paris-Sud
http://www.math.u-psud.fr/∼santambr/

Grenoble, October 4th, 2013,
Modelisation with optimal transport

1. **Monotone trasports**
   - The 1D monotone transport
   - The Brenier map, gradient of a convex potential
   - The Knothe-Rosenblatt map
2. **Convergence as $t \to 0$ for the cost $|x_1 - y_1|^2 + t|x_2 - y_2|^2$**
   - A conjecture by Y. Brenier
   - A proof in the spirit of $\Gamma-$developments
   - Assumptions and counter-examples
   - Atoms in the disintegrated measures
3. **Dynamics as t moves in the semi-discrete case**
   - An ODE for the potential
   - Evolution of cells
4. **Dynamics in the continuous case**
   - The PDE for the potential
   - The initial condition
   - Well-posedness
   - Numerical solution

1. **Monotone trasports**
   - The 1D monotone transport
   - The Brenier map, gradient of a convex potential
   - The Knothe-Rosenblatt map
2. **Convergence as $t \to 0$ for the cost $|x_1 - y_1|^2 + t|x_2 - y_2|^2$**
   - A conjecture by Y. Brenier
   - A proof in the spirit of $\Gamma-$developments
   - Assumptions and counter-examples
   - Atoms in the disintegrated measures
3. **Dynamics as t moves in the semi-discrete case**
   - An ODE for the potential
   - Evolution of cells
4. **Dynamics in the continuous case**
   - The PDE for the potential
   - The initial condition
   - Well-posedness
   - Numerical solution

1. **Monotone trasports**
   - The 1D monotone transport
   - The Brenier map, gradient of a convex potential
   - The Knothe-Rosenblatt map
2. **Convergence as $t \to 0$ for the cost $|x_1 - y_1|^2 + t|x_2 - y_2|^2$**
   - A conjecture by Y. Brenier
   - A proof in the spirit of $\Gamma-$developments
   - Assumptions and counter-examples
   - Atoms in the disintegrated measures
3. **Dynamics as t moves in the semi-discrete case**
   - An ODE for the potential
   - Evolution of cells
4. **Dynamics in the continuous case**
   - The PDE for the potential
   - The initial condition
   - Well-posedness
   - Numerical solution

1. **Monotone trasports**
   - The 1D monotone transport
   - The Brenier map, gradient of a convex potential
   - The Knothe-Rosenblatt map
2. **Convergence as $t \to 0$ for the cost $|x_1 - y_1|^2 + t|x_2 - y_2|^2$**
   - A conjecture by Y. Brenier
   - A proof in the spirit of $\Gamma-$developments
   - Assumptions and counter-examples
   - Atoms in the disintegrated measures
3. **Dynamics as t moves in the semi-discrete case**
   - An ODE for the potential
   - Evolution of cells
4. **Dynamics in the continuous case**
   - The PDE for the potential
   - The initial condition
   - Well-posedness
   - Numerical solution

Monotone transports

1D, Brenier, Knothe

Very briefly, something you all know about the optimal transport problem

Monge Problem : $\min \int c(x, T(x))\mu(dx) \ : \ T_{\#}\mu = \nu$
proposed by G. Monge in 1781, for $c(x, y) = |x - y|$.

Kantorovich Problem : (1942) $\min \int c(x, y) d\gamma \ : \ \gamma \in \Pi(\mu, \nu)$
where $\Pi(\mu, \nu) := \{\gamma \ : \ (\pi_x)_\sharp \gamma = \mu, \ (\pi_y)_\sharp \gamma = \nu\}$.
This gives again Monge's framework when $\gamma = (id \times T)_{\#}\mu$.

Advantages of Kantorovich's formulation

- it's a convex problem
- it always has a solution (if c is l.s.c.)
- il has a dual formulation :

$$\min \int c \, d\gamma = \sup \int \phi d\mu + \int \psi d\nu \ : \ \phi(x) + \psi(y) \leq c(x, y).$$

Very briefly, something you all know about the optimal transport problem

Monge Problem : $\min \int c(x, T(x))\mu(dx) \ : \ T_{\#}\mu = \nu$
proposed by G. Monge in 1781, for $c(x, y) = |x - y|$.

Kantorovich Problem : (1942) $\min \int c(x, y) d\gamma \ : \ \gamma \in \Pi(\mu, \nu)$
where $\Pi(\mu, \nu) := \{\gamma \ : \ (\pi_x)_\sharp \gamma = \mu, \ (\pi_y)_\sharp \gamma = \nu\}$.
This gives again Monge's framework when $\gamma = (id \times T)_{\#}\mu$.

Advantages of Kantorovich's formulation

- it's a convex problem
- it always has a solution (if c is l.s.c.)
- il has a dual formulation :

$$\min \int c \, d\gamma = \sup \int \phi d\mu + \int \psi d\nu \ : \ \phi(x) + \psi(y) \leq c(x, y).$$

The monotone transport in 1D

Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, if μ has no atoms, there exists unique an increasing map $T : \mathbb{R} \to \mathbb{R}$ such that $T_{\#}\mu = \nu$.

If F and G are the cumulative distribution functions of μ and ν, respectively, and if G is strictly increasing on $\operatorname{spt} \nu$ (i.e. if $\operatorname{spt} \nu$ is an interval), we can compute it through $T = G^{-1} \circ F$ (if ν has not full support a generalized inverse of G should be used).

This map turns out to be optimal for all the costs of the form $c(x, y) = h(x - y)$ with h convex (and it is the unique optimizer if h is strictly convex). In particular, this covers the quadratic case $c(x, y) = |x - y|^2$.

It is very easy to compute, but only works in 1D.

The monotone transport in 1D

Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, if μ has no atoms, there exists unique an increasing map $T : \mathbb{R} \to \mathbb{R}$ such that $T_{\#}\mu = \nu$.

If F and G are the cumulative distribution functions of μ and ν, respectively, and if G is strictly increasing on $\operatorname{spt}\nu$ (i.e. if $\operatorname{spt}\nu$ is an interval), we can compute it through $T = G^{-1} \circ F$ (if ν has not full support a generalized inverse of G should be used).

This map turns out to be optimal for all the costs of the form $c(x, y) = h(x - y)$ with h convex (and it is the unique optimizer if h is strictly convex). In particular, this covers the quadratic case $c(x, y) = |x - y|^2$.

It is very easy to compute, but only works in 1D.

The monotone transport in 1D

Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, if μ has no atoms, there exists unique an increasing map $T : \mathbb{R} \to \mathbb{R}$ such that $T_\# \mu = \nu$.

If F and G are the cumulative distribution functions of μ and ν, respectively, and if G is strictly increasing on $\operatorname{spt} \nu$ (i.e. if $\operatorname{spt} \nu$ is an interval), we can compute it through $T = G^{-1} \circ F$ (if ν has not full support a generalized inverse of G should be used).

This map turns out to be optimal for all the costs of the form $c(x, y) = h(x - y)$ with h convex (and it is the unique optimizer if h is strictly convex). In particular, this covers the quadratic case $c(x, y) = |x - y|^2$.

It is very easy to compute, but only works in 1D.

The quadratic cost in $\mathbb{R}^d$

If $d > 1$ and μ, ν are measures on $\Omega \subset \mathbb{R}^d$ the situation is trickier but Brenier proved the following : if μ is nice (for instance $\mu \ll \mathcal{L}^d$), then there exists unique an optimal map, and it given by $T = \nabla\phi$, with ϕ convex.

It has some monotonicity property (for instance, DT is a symmetric and positive definite matrix). But it is trickier to compute. The change-of-variable-formula, if $\mu = f(x)dx$ and $\nu = g(y)dy$, gives the Jacobian condition $\det DT = \frac{f}{g\circ T}$, which reads here

$$\det(D^2\phi) = \frac{f}{g \circ \nabla\phi}, \qquad \text{with } \phi \text{ convex},$$

(Monge-Ampère equation). Its "boundary" condition is given by $\nabla\phi(x) \in \Omega$ for all $x \in \Omega$. This PDE is nonlinear and difficult to solve, both numerically and theoretically.

Some regularity theorems exist giving $\phi \in C^{k+2,\alpha}$ if f, g are bounded from below and belong to $C^{k,\alpha}$ and $\operatorname{spt}\nu$ is convex.. In this case T is $C^{k+1,\alpha}$.

Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, *CPAM*, 1991.

The quadratic cost in $\mathbb{R}^d$

If $d > 1$ and μ, ν are measures on $\Omega \subset \mathbb{R}^d$ the situation is trickier but Brenier proved the following : if μ is nice (for instance $\mu \ll \mathcal{L}^d$), then there exists unique an optimal map, and it given by $T = \nabla\phi$, with ϕ convex.

It has some monotonicity property (for instance, DT is a symmetric and positive definite matrix). But it is trickier to compute. The change-of-variable-formula, if $\mu = f(x)dx$ and $\nu = g(y)dy$, gives the Jacobian condition $\det DT = \frac{f}{g\circ T}$, which reads here

$$\det(D^2\phi) = \frac{f}{g \circ \nabla\phi}, \quad \text{with } \phi \text{ convex},$$

(Monge-Ampère equation). Its "boundary" condition is given by $\nabla\phi(x) \in \Omega$ for all $x \in \Omega$. This PDE is nonlinear and difficult to solve, both numerically and theoretically.

Some regularity theorems exist giving $\phi \in C^{k+2,\alpha}$ if f, g are bounded from below and belong to $C^{k,\alpha}$ and $\operatorname{spt}\nu$ is convex.. In this case T is $C^{k+1,\alpha}$.

Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, *CPAM*, 1991.

The Knothe-Rosenblatt rearrangement

Here is another reasonable transport (increasing for the lexicographic order) : there exists unique a map T_K of the form

$$T_K(x_1, x_2, \dots, x_d) := (T^1(x_1), T^2(x_1, x_2), \dots, T^d(x_1, x_2, \dots, x_d))$$

where all the $T^i(x_1, x_2, \dots, x_{i-1}, \cdot)$ are increasing, sending μ onto ν.

Recursive construction : If $d = 1$ just take the monotone map. If $d > 1$, let μ_1 and ν_1 be the projections on of μ and ν on the first variable and T^1 be the monotone map between them. Then, disintegrate μ and ν according to the first variable, and define $(T^2, T^3, \dots, T^d)(x_1, \cdot, \dots, \cdot)$ as the Knothe transport in dimension $(d-1)$ between μ_{x_1} and $\nu_{T^1(x_1)}$.

T_K is much easier to compute than the Brenier map. Yet, it is not optimal, and its definition is anisotropic.

Regularity : T_K has the same regularity of the densities, not more. Its Jacobian DT_K is triangular, with positive coefficients on the diagonal.

H. Knothe, Contributions to the theory of convex bodies, *MI Math. J.* 1957
M. Rosenblatt, Remarks on a multivariate transformation, *Ann. Math. Stat.*, 1952.

The Knothe-Rosenblatt rearrangement

Here is another reasonable transport (increasing for the lexicographic order) : there exists unique a map T_K of the form

$$T_K(x_1, x_2, \ldots, x_d) := (T^1(x_1), T^2(x_1, x_2), \ldots, T^d(x_1, x_2, \ldots, x_d))$$

where all the $T^i(x_1, x_2, \ldots, x_{i-1}, \cdot)$ are increasing, sending μ onto ν.

Recursive construction : If $d = 1$ just take the monotone map. If $d > 1$, let μ_1 and ν_1 be the projections on of μ and ν on the first variable and T^1 be the monotone map between them. Then, disintegrate μ and ν according to the first variable, and define $(T^2, T^3, \ldots, T^d)(x_1, \cdot, \ldots, \cdot)$ as the Knothe transport in dimension $(d-1)$ between μ_{x_1} and $\nu_{T^1(x_1)}$.

T_K is much easier to compute than the Brenier map. Yet, it is not optimal, and its definition is anisotropic.

Regularity : T_K has the same regularity of the densities, not more. Its Jacobian DT_K is triangular, with positive coefficients on the diagonal.

H. Knothe, Contributions to the theory of convex bodies, *MI Math. J.* 1957
M. Rosenblatt, Remarks on a multivariate transformation, *Ann. Math. Stat.*, 1952.

The Knothe-Rosenblatt rearrangement

Here is another reasonable transport (increasing for the lexicographic order) : there exists unique a map T_K of the form

$$T_K(x_1, x_2, \dots, x_d) := (T^1(x_1), T^2(x_1, x_2), \dots, T^d(x_1, x_2, \dots, x_d))$$

where all the $T^i(x_1, x_2, \dots, x_{i-1}, \cdot)$ are increasing, sending μ onto ν.

Recursive construction : If $d = 1$ just take the monotone map. If $d > 1$, let μ_1 and ν_1 be the projections on of μ and ν on the first variable and T^1 be the monotone map between them. Then, disintegrate μ and ν according to the first variable, and define $(T^2, T^3, \dots, T^d)(x_1, \cdot, \dots, \cdot)$ as the Knothe transport in dimension $(d-1)$ between μ_{x_1} and $\nu_{T^1(x_1)}$.

T_K is much easier to compute than the Brenier map. Yet, it is not optimal, and its definition is anisotropic.

Regularity : T_K has the same regularity of the densities, not more. Its Jacobian DT_K is triangular, with positive coefficients on the diagonal.

H. KNOTHE, Contributions to the theory of convex bodies, *MI Math. J.* 1957
M. ROSENBLATT, Remarks on a multivariate transformation, *Ann. Math. Stat.*, 1952.

Convergence of quadratic costs to Knothe

The cost $|x_1 - y_1|^2 + t|x_2 - y_2|^2$ as $t \to 0$

A reasonable conjecture

Let us consider the weighted quadratic cost

$$c_t(x,y) := \sum_{i=1}^{d} t^{i-1}|x_i - y_i|^2.$$

If $\mu \ll \mathcal{L}^d$, the corresponding optimal transportation problem admits a unique solution T_t. According to a **conjecture by Y. Brenier**, it is natural to expect the convergence of T_t to the Knothe transport T_K.

Why ? because as $t \to 0$ the main criterion becomes the minimization of the cost $|x_1 - y_1|^2$. This selects T^1 but gives nothing on the other variables. We pass to the second most important criterion : minimizing $|x_2 - y_2|^2$, and this provides T^2. And we go on.

This is **in the same spirit of a** $\Gamma-$**convergence development** $c_t = c^1 + tc^2 + t^2c^3 + \dots$. If c^1 has not a unique minimizer, we select the one that also minimizes c^2 among minimizers of c^1. And if it has not uniqueness neither, we look at $c^3\dots$

An example

Let's see a case where explicit solutions are available. Take $d = 2$, and μ and ν two Gaussian measures where

$$\mu = N(0, \mathrm{Id}) \quad \text{and} \quad \nu = N\left(0, \begin{pmatrix} a & b \\ b & c \end{pmatrix}\right)$$

(with $ac > b^2$, $a > 0$). We can check that T_t is linear with matrix

$$T_t = \frac{1}{\sqrt{a + ct^2 + 2t\sqrt{ac - b^2}}} \begin{pmatrix} a + t\sqrt{ac - b^2} & bt \\ b & ct + \sqrt{ac - b^2} \end{pmatrix}$$

which converges as $t \to 0$ to

$$\begin{pmatrix} \sqrt{a} & 0 \\ b/\sqrt{a} & \sqrt{c - b^2/a} \end{pmatrix}$$

which is precisely the matrix of the Knothe transport from μ to ν.

A theorem

Assumption *(H-source)* **:** the measure μ^1, as well as μ^1-almost all the measures $\mu^2_{x_1}$, and the measures $\mu^3_{x_1,x_2}$...up to almost all the measures $\mu^d_{x_1,x_2,\dots,x_{d-1}}$, which are all measures on the real line, have no atoms.
Assumption *(H-target)* **:** the measure ν^1, as well as ν^1-almost all the measures $\nu^2_{x_1}$, and the measures $\nu^3_{x_1,x_2}$...up to almost all the measures $\nu^{d-1}_{x_1,x_2,\dots,x_{d-2}}$, have no atoms neither.

Theorem

Let μ and ν satisfy (H-source) and (H-target), γ_t be an optimal plan for the costs $c_t(x,y)$, T_K the Knothe-Rosenblatt map between μ and ν and γ_K the associated transport plan. Then $\gamma_t \rightharpoonup \gamma_K$ as $t \to 0$.
Moreover, should the plans γ_t be induced by transport maps T_t, then these maps would converge to T_K in $L^2(\mu)$ as $t \to 0$.

G. Carlier, A. Galichon , F. Santambrogio, From Knothe's transport to Brenier's map and a continuation method for optimal transport, *SIAM J. Math. An.*, 2010

Atoms

Counter-example. Surprisingly, the absence of atoms in ν is really necessary. Look at this example in $[-1,1]\times[-1,1]\subset\mathbb{R}^2$ where

$$\mu=\frac{1}{2}1_{\{x_1x_2<0\}}dx \quad \text{and} \quad \nu=\frac{1}{2}\mathcal{H}^1_{|S} \text{ with } S=\{0\}\times[-1,1].$$

The Knothe-Rosenblatt map is $T_K(x):=(0,2x_1+sgn(x_2)))$. The optimal transport for each cost c_t is $T_t(x):=(0,x_1)$ (no transport may do better than this one, which projects on the support of ν). The reason for the lack of convergence is the atom in the measure $\nu^1=\delta_0$.

Don't despair ! This means that we cannot apply the result if ν itself is purely atomic...yet, looking at the proof we can also deal with the following case. Keep *(H-source)* on μ but suppose that ν is concentrated on a set S with the property

$$y,z\in S,\quad y\neq z\Rightarrow y_1\neq z_1.$$

This allows to deal with almost all finite atomic measures ν.

Atoms

Counter-example. Surprisingly, the absence of atoms in ν is really necessary. Look at this example in $[-1,1]\times[-1,1]\subset\mathbb{R}^2$ where

$$\mu = \frac{1}{2} 1_{\{x_1 x_2 < 0\}} dx \quad \text{and} \quad \nu = \frac{1}{2}\mathcal{H}^1_{|S} \text{ with } S = \{0\}\times[-1,1].$$

The Knothe-Rosenblatt map is $T_K(x) := (0, 2x_1 + sgn(x_2)))$. The optimal transport for each cost c_t is $T_t(x) := (0, x_1)$ (no transport may do better than this one, which projects on the support of ν). The reason for the lack of convergence is the atom in the measure $\nu^1 = \delta_0$.

Don't despair ! This means that we cannot apply the result if ν itself is purely atomic…yet, looking at the proof we can also deal with the following case. Keep *(H-source)* on μ but suppose that ν is concentrated on a set S with the property

$$y, z \in S, \quad y \neq z \Rightarrow y_1 \neq z_1.$$

This allows to deal with almost all finite atomic measures ν.

Semi-discrete evolution

An ODE for the potential

From Knothe to Brenier

The Knothe transport T_K is easy to compute because it is essentially 1D ; the Brenier map is the optimal map for c_1 ; the optimal maps for c_t converge to T_K as $t \to 0$.

Idea : can we start from T_K and let t improve from 0 to 1 in order to compute T_1 ?

Let us start from the semidiscrete case, i.e. μ is a smooth density on Ω and ν is a finite atomic measure with N atoms, say μ uniform on some convex polyhedron $\Omega \subset \mathbb{R}^2$ and $\nu = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ (where all the points y_i have a different first coordinate $y_i^{(2)}$).

The transport map, piecewise constant on some unknown Voronoi-type cells, can be computed from the **potential in the dual problem**.

The dual problem reads

$$\sup_p \quad \Phi(p,t) := \frac{1}{N}\sum_{i=1}^N p_i + \int_\Omega p_t^*(x)dx,$$

where $p_t^*(x) = \min_i\{c_t(x,y_i) - p_i\}$ and we set $p_1 = 0$. For each t, there is a unique maximizer $p(t)$. It belongs to $\mathbb{R}^N$ and we look for its evolution.

From Knothe to Brenier

The Knothe transport T_K is easy to compute because it is essentially 1D ; the Brenier map is the optimal map for c_1 ; the optimal maps for c_t converge to T_K as $t \to 0$.

Idea : can we start from T_K and let t improve from 0 to 1 in order to compute T_1 ?

Let us start from the semidiscrete case, i.e. μ is a smooth density on Ω and ν is a finite atomic measure with N atoms, say μ uniform on some convex polyhedron $\Omega \subset \mathbb{R}^2$ and $\nu = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ (where all the points y_i have a different first coordinate $y_i^{(2)}$).

The transport map, piecewise constant on some unknown Voronoi-type cells, can be computed from the **potential in the dual problem**.

The dual problem reads

$$\sup_p \quad \Phi(p,t) := \frac{1}{N}\sum_{i=1}^N p_i + \int_\Omega p_t^*(x)dx,$$

where $p_t^*(x) = \min_i\{c_t(x,y_i) - p_i\}$ and we set $p_1 = 0$. For each t, there is a unique maximizer $p(t)$. It belongs to $\mathbb{R}^N$ and we look for its evolution.

From Knothe to Brenier

The Knothe transport T_K is easy to compute because it is essentially 1D ; the Brenier map is the optimal map for c_1 ; the optimal maps for c_t converge to T_K as $t \to 0$.

Idea : can we start from T_K and let t improve from 0 to 1 in order to compute T_1 ?

Let us start from the semidiscrete case, i.e. μ is a smooth density on Ω and ν is a finite atomic measure with N atoms, say μ uniform on some convex polyhedron $\Omega \subset \mathbb{R}^2$ and $\nu = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ (where all the points y_i have a different first coordinate $y_i^{(2)}$).

The transport map, piecewise constant on some unknown Voronoi-type cells, can be computed from the **potential in the dual problem**.

The dual problem reads

$$\sup_p \quad \Phi(p,t) := \frac{1}{N}\sum_{i=1}^N p_i + \int_\Omega p_t^*(x)dx,$$

where $p_t^*(x) = \min_i\{c_t(x,y_i) - p_i\}$ and we set $p_1 = 0$. For each t, there is a unique maximizer $p(t)$. It belongs to $\mathbb{R}^N$ and we look for its evolution.

The ODE

For each (p,t), set $C(p,t)_i = \{x \in \Omega : \inf_j c_t(x,y_j) - p_j = c_t(x,y_i) - p_i\}$. The function $\Phi(.,t)$ is concave differentiable and its gradient is given by

$$\frac{\partial \Phi_t}{\partial p_i}(p,t) = \frac{1}{N} - |C(p,t)_i|.$$

By concavity, the maximizer $p(t)$ is characterized by $\nabla \Phi_t(p(t),t) = 0$. Differentiating, we obtain a differential equation for the evolution of $p(t)$:

$$\frac{\partial}{\partial t}\nabla_p \Phi(p(t),t) + D^2_{p,p}\Phi(p(t),t) \cdot \frac{dp}{dt}(t) = 0.$$

All the quantities we are interested in depend on the position of the vertices of the cells $C(p,t)_i$, which are all polygons.

Result : The positions of these vertices depend in a Lipschitz way on p and t ; the matrix $D^2_{p,p}\Phi(p(t),t)$ is invertible in a suitable domain ; we can apply Cauchy-Lipschitz theorem to the ODE

$$\frac{dp}{dt}(t) = -D^2_{p,p}\Phi(p(t),t)^{-1}\left(\frac{\partial}{\partial t}\nabla_p \Phi(p(t),t)\right)$$

The ODE

For each (p,t), set $C(p,t)_i = \{x \in \Omega \,:\, \inf_j c_t(x,y_j) - p_j = c_t(x,y_i) - p_i\}$. The function $\Phi(.,t)$ is concave differentiable and its gradient is given by

$$\frac{\partial \Phi_t}{\partial p_i}(p,t) = \frac{1}{N} - |C(p,t)_i|.$$

By concavity, the maximizer $p(t)$ is characterized by $\nabla\Phi_t(p(t),t) = 0$. Differentiating, we obtain a differential equation for the evolution of $p(t)$:

$$\frac{\partial}{\partial t}\nabla_p \Phi(p(t),t) + D^2_{p,p}\Phi(p(t),t)\cdot \frac{dp}{dt}(t) = 0.$$

All the quantities we are interested in depend on the position of the vertices of the cells $C(p,t)_i$, which are all polygons.

Result : The positions of these vertices depend in a Lipschitz way on p and t ; the matrix $D^2_{p,p}\Phi(p(t),t)$ is invertible in a suitable domain ; we can apply Cauchy-Lipschitz theorem to the ODE

$$\frac{dp}{dt}(t) = -D^2_{p,p}\Phi(p(t),t)^{-1}\left(\frac{\partial}{\partial t}\nabla_p \Phi(p(t),t)\right)$$

The ODE

For each (p,t), set $C(p,t)_i = \{x \in \Omega \,:\, \inf_j c_t(x,y_j) - p_j = c_t(x,y_i) - p_i\}$. The function $\Phi(.,t)$ is concave differentiable and its gradient is given by

$$\frac{\partial \Phi_t}{\partial p_i}(p,t) = \frac{1}{N} - |C(p,t)_i|.$$

By concavity, the maximizer $p(t)$ is characterized by $\nabla \Phi_t(p(t),t) = 0$. Differentiating, we obtain a differential equation for the evolution of $p(t)$:

$$\frac{\partial}{\partial t} \nabla_p \Phi(p(t),t) + D^2_{p,p}\Phi(p(t),t) \cdot \frac{dp}{dt}(t) = 0.$$

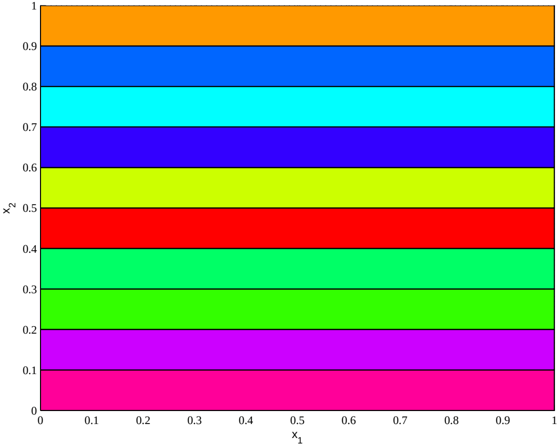
All the quantities we are interested in depend on the position of the vertices of the cells $C(p,t)_i$, which are all polygons.

Result : The positions of these vertices depend in a Lipschitz way on p and t ; the matrix $D^2_{p,p}\Phi(p(t),t)$ is invertible in a suitable domain ; we can apply Cauchy-Lipschitz theorem to the ODE

$$\frac{dp}{dt}(t) = -D^2_{p,p}\Phi(p(t),t)^{-1} \left(\frac{\partial}{\partial t} \nabla_p \Phi(p(t),t) \right)$$

Cells evolution

Different shapes of the cells in a simple semi-discrete case for $t \in [0, +\infty[$.



Continuous evolution

A PDE for the potential

Monge-Ampère equation

In $\mathbb{R}^d$, take the matrix

$$A_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

and the cost $c_t(x,y) = \frac{1}{2} A_t(x-y)\cdot(x-y)$. The optimal transport is given by $T_t(x) = x - A_t^{-1}\nabla\phi_t$. The MA equation gives

$$\det(\mathrm{Id} - A_t^{-1}D^2\phi_t) = \frac{f}{g(x - A_t^{-1}\nabla\phi_t(x))}.$$

Let us take the easiest case, i.e. $g = 1$ and let's differentiate w.r.t. t :

$$\mathrm{trace}\left[(A_t - D^2\phi_t)^{-1}D^2\phi_t'\right] = -\mathrm{trace}\left[(\mathrm{Id} - A_t^{-1}D^2\phi_t)^{-1}\left(\frac{d}{dt}(A_t)^{-1}\right)D^2\phi_t\right].$$

The equation is therefore

$$\frac{\partial\phi_t}{\partial t} = \chi \quad \text{with} \quad \mathrm{trace}\left[(\mathrm{Id} - A_t^{-1}D^2\phi_t)^{-1}D^2\chi\right] = h(t, D^2\phi_t).$$

Monge-Ampère equation

In $\mathbb{R}^d$, take the matrix

$$A_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

and the cost $c_t(x, y) = \frac{1}{2} A_t(x - y) \cdot (x - y)$. The optimal transport is given by $T_t(x) = x - A_t^{-1} \nabla \phi_t$. The MA equation gives

$$\det(\mathrm{Id} - A_t^{-1} D^2 \phi_t) = \frac{f}{g(x - A_t^{-1} \nabla \phi_t(x))}.$$

Let us take the easiest case, i.e. $g = 1$ and let's differentiate w.r.t. t :

$$\mathrm{trace}\left[(A_t - D^2\phi_t)^{-1} D^2 \phi_t'\right] = -\mathrm{trace}\left[(\mathrm{Id} - A_t^{-1} D^2 \phi_t)^{-1} \left(\frac{d}{dt}(A_t)^{-1}\right) D^2 \phi_t\right].$$

The equation is therefore

$$\frac{\partial \phi_t}{\partial t} = \chi \quad \text{with} \quad \mathrm{trace}\left[(\mathrm{Id} - A_t^{-1} D^2 \phi_t)^{-1} D^2 \chi\right] = h(t, D^2 \phi_t).$$

Initial condition

What about $\lim_{t\to 0} \phi_t$? Unfortunately, the limit potential is that associated to the cost $\frac{1}{2}|x_1 - y_1|^2$, i.e. it only depends on the measures μ_1 and ν_1. In particular, there is no hope for a uniqueness result.

Good idea Write $\phi_t = u_t(x_1) + tv_t(x_1, x_2)$ (in higher dimension we put $+t^2 w_t(x_1, x_2, x_3) \dots$). This allows to

- give initial conditions : u_0 is the potential between μ_1 and ν_1 and, for each x_1, the function $v_0(x_1, \cdot)$ is the potential between μ_{x_1} and ν_{y_1} with $y_1 = T^1(x_1)$;
- de-singularize the equation, since

$$A_t^{-1} D^2\phi = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} \partial_{11} u + t\partial_{11} v & t\partial_{12} v \\ t\partial_{11} v & t\partial_{22} v \end{pmatrix} = \begin{pmatrix} \partial_{11} u + t\partial_{11} v & t\partial_{12} v \\ \partial_{11} v & \partial_{22} v \end{pmatrix}$$

N. Bonnotte, From Knothe's Rearrangement to Brenier's Optimal Transport map, *SIAM J. Math. An.*, 2013 ; + cotutelle PhD thesis at Orsay and SNS Pisa, to be defended soon

Initial condition

What about $\lim_{t\to 0}\phi_t$? Unfortunately, the limit potential is that associated to the cost $\frac{1}{2}|x_1-y_1|^2$, i.e. it only depends on the measures μ_1 and ν_1. In particular, there is no hope for a uniqueness result.

Good idea Write $\phi_t = u_t(x_1) + tv_t(x_1, x_2)$ (in higher dimension we put $+t^2 w_t(x_1, x_2, x_3)\dots$). This allows to

- give initial conditions : u_0 is the potential between μ_1 and ν_1 and, for each x_1, the function $v_0(x_1,\cdot)$ is the potential between μ_{x_1} and ν_{y_1} with $y_1 = T^1(x_1)$;
- de-singularize the equation, since

$$A_t^{-1}D^2\phi = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1}\end{pmatrix}\cdot\begin{pmatrix} \partial_{11}u + t\partial_{11}v & t\partial_{12}v \\ t\partial_{11}v & t\partial_{22}v\end{pmatrix} = \begin{pmatrix} \partial_{11}u + t\partial_{11}v & t\partial_{12}v \\ \partial_{11}v & \partial_{22}v\end{pmatrix}$$

N. Bonnotte, From Knothe's Rearrangement to Brenier's Optimal Transport map, *SIAM J. Math. An.*, 2013 ; + cotutelle PhD thesis at Orsay and SNS Pisa, to be defended soon

Initial condition

What about $\lim_{t\to 0}\phi_t$? Unfortunately, the limit potential is that associated to the cost $\frac{1}{2}|x_1-y_1|^2$, i.e. it only depends on the measures μ_1 and ν_1. In particular, there is no hope for a uniqueness result.

Good idea Write $\phi_t = u_t(x_1) + tv_t(x_1,x_2)$ (in higher dimension we put $+t^2 w_t(x_1,x_2,x_3)\dots$). This allows to

- give initial conditions : u_0 is the potential between μ_1 and ν_1 and, for each x_1, the function $v_0(x_1,\cdot)$ is the potential between μ_{x_1} and ν_{y_1} with $y_1 = T^1(x_1)$;
- de-singularize the equation, since

$$A_t^{-1}D^2\phi = \begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\begin{pmatrix}\partial_{11}u + t\partial_{11}v & t\partial_{12}v\\ t\partial_{11}v & t\partial_{22}v\end{pmatrix} = \begin{pmatrix}\partial_{11}u + t\partial_{11}v & t\partial_{12}v\\ \partial_{11}v & \partial_{22}v\end{pmatrix}$$

N. Bonnotte, From Knothe's Rearrangement to Brenier's Optimal Transport map, *SIAM J. Math. An.*, 2013 ; + cotutelle PhD thesis at Orsay and SNS Pisa, to be defended soon

Initial condition

What about $\lim_{t\to 0}\phi_t$? Unfortunately, the limit potential is that associated to the cost $\frac{1}{2}|x_1 - y_1|^2$, i.e. it only depends on the measures μ_1 and ν_1. In particular, there is no hope for a uniqueness result.

Good idea Write $\phi_t = u_t(x_1) + tv_t(x_1, x_2)$ (in higher dimension we put $+t^2 w_t(x_1, x_2, x_3)\dots$). This allows to

- give initial conditions : u_0 is the potential between μ_1 and ν_1 and, for each x_1, the function $v_0(x_1,\cdot)$ is the potential between μ_{x_1} and ν_{y_1} with $y_1 = T^1(x_1)$;
- de-singularize the equation, since

$$A_t^{-1}D^2\phi = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} \partial_{11}u + t\partial_{11}v & t\partial_{12}v \\ t\partial_{11}v & t\partial_{22}v \end{pmatrix} = \begin{pmatrix} \partial_{11}u + t\partial_{11}v & t\partial_{12}v \\ \partial_{11}v & \partial_{22}v \end{pmatrix}$$

N. Bonnotte, From Knothe's Rearrangement to Brenier's Optimal Transport map, *SIAM J. Math. An.*, 2013 ; + cotutelle PhD thesis at Orsay and SNS Pisa, to be defended soon

Well-posedness

For $t > 0$, an implicit function theorem in the space $\mathbb{R} \times C^{2,\alpha}(x_1) \times C^{2,\alpha}(x_1, x_2)$ applied to the function $(t, u, v) \mapsto \det(I - A_t^{-1} D^2(u + tv))$ allows to prove well-posedness of the equation.

Problem : for $t = 0$ there is a **loss of regularity** : u_t, v_t have two extra derivatives w.r.t. f, while v_0 has the same regularity in x_1 as f. No space $C^{k\alpha}$ is suitable for this IFT.
Solution : we must choose the space C^∞ and use the IFT by **Nash-Moser**. Nicolas worked hard on that, and proved (on the torus, to avoid boundary issues) that it works !

Notice that, besides the theoretical speculations, the equation is not so bad, and suggests that an explicit method can be used to solve it.

Well-posedness

For $t > 0$, an implicit function theorem in the space $\mathbb{R} \times C^{2,\alpha}(x_1) \times C^{2,\alpha}(x_1, x_2)$ applied to the function $(t, u, v) \mapsto \det(I - A_t^{-1} D^2(u + tv))$ allows to prove well-posedness of the equation.

Problem : for $t = 0$ there is a **loss of regularity** : u_t, v_t have two extra derivatives w.r.t. f, while v_0 has the same regularity in x_1 as f. No space $C^{k\alpha}$ is suitable for this IFT.

Solution : we must choose the space C^∞ and use the IFT by **Nash-Moser**. Nicolas worked hard on that, and proved (on the torus, to avoid boundary issues) that it works !

Notice that, besides the theoretical speculations, the equation is not so bad, and suggests that an explicit method can be used to solve it.

Well-posedness

For $t > 0$, an implicit function theorem in the space $\mathbb{R} \times C^{2,\alpha}(x_1) \times C^{2,\alpha}(x_1, x_2)$ applied to the function $(t, u, v) \mapsto \det(I - A_t^{-1} D^2(u + tv))$ allows to prove well-posedness of the equation.

Problem : for $t = 0$ there is a **loss of regularity** : u_t, v_t have two extra derivatives w.r.t. f, while v_0 has the same regularity in x_1 as f. No space $C^{k\alpha}$ is suitable for this IFT.
Solution : we must choose the space C^∞ and use the IFT by **Nash-Moser**. Nicolas worked hard on that, and proved (on the torus, to avoid boundary issues) that it works !

Notice that, besides the theoretical speculations, the equation is not so bad, and suggests that an explicit method can be used to solve it.

Well-posedness

For $t > 0$, an implicit function theorem in the space $\mathbb{R} \times C^{2,\alpha}(x_1) \times C^{2,\alpha}(x_1, x_2)$ applied to the function $(t, u, v) \mapsto \det(I - A_t^{-1} D^2(u + tv))$ allows to prove well-posedness of the equation.

Problem : for $t = 0$ there is a **loss of regularity** : u_t, v_t have two extra derivatives w.r.t. f, while v_0 has the same regularity in x_1 as f. No space $C^{k\alpha}$ is suitable for this IFT.
Solution : we must choose the space C^∞ and use the IFT by **Nash-Moser**. Nicolas worked hard on that, and proved (on the torus, to avoid boundary issues) that it works !

Notice that, besides the theoretical speculations, the equation is not so bad, and suggests that an explicit method can be used to solve it.

Some numerical pictures – Knothe

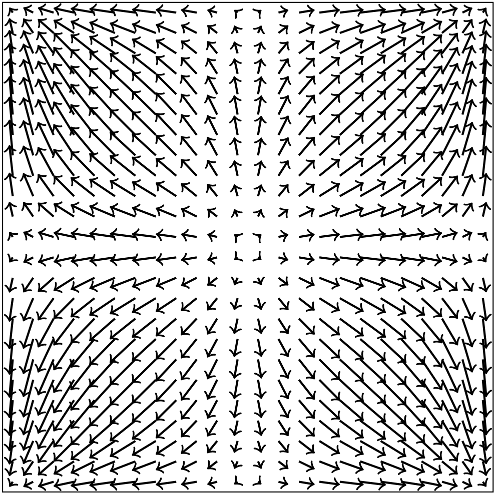


FIGURE: The Knothe–Rosenblatt rearrangement.

Some numerical pictures – Knothe 2

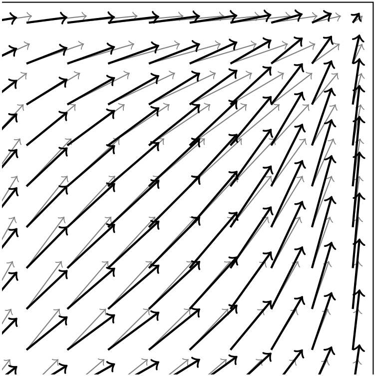


FIGURE: The black arrows represent the Knothe–Rosenblatt rearrangement, and the gray ones its symmetric. The discrepancy comes from the fact that the rearrangement' is anisotropic.

Some numerical pictures – computation of the optimal map

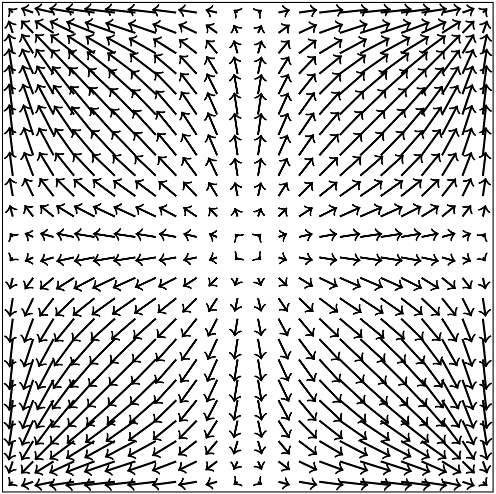


FIGURE: Computation of Brenier's optimal map by the evolution $u_t + tv_t$.

Some numerical pictures – comparison Knothe-Brenier

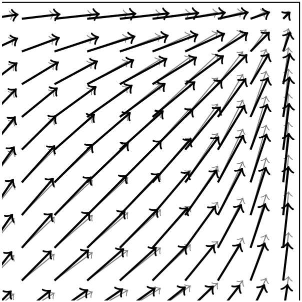


FIGURE: The black arrows represent Brenier's optimal transport map, and the gray ones the Knothe–Rosenblatt rearrangement.

Here it is,

Thanks for your attention